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IMPROVEMENTS AND EXTENSIONS TO THE MILLER-TUCKER-ZEMLIN SUBTOUR ELIMINATION CONSTRAINTS*

by

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Abstract

This paper shows how the subtour elimination constraints developed by Miller, Tucker and Zemlin for the traveling salesman problem can be improved and extended to various types of vehicle routing problems.

Key words

Subtour elimination, vehicle routing, lifting.

Résumé

On montre que les contraintes d'élimination de sous-tours proposées par Miller, Tucker et Zemlin pour le problème du voyageur de commerce peuvent être améliorées et généralisées à divers problèmes de fabrication de tournées.

Mots clés

Contraintes d'élimination de sous-tours, fabrication de tournées, lifting.

1. Introduction

Consider a graph $G = (V, A)$ where $V = \{v_1, \ldots, v_n\}$ is the vertex set and $A = \{(v_i, v_j) : v_i, v_j \in V, i \neq j\}$ is the arc set. Let $C = (c_{ij})$ be a vector of positive costs, distances, or travel times associated with A. The *traveling salesman problem* (TSP) consists of determining a least cost Hamiltonian circuit on G. Several types of *vehicle routing problems* (VRPs) can also be defined on G. Let v_1 represent a *depot* at which m identical vehicles are based. VRPs consist of determining m routes of least total cost from the depot, in such a way that every vertex of $V\backslash\{v_1\}$ is visited exactly once and that some side constraints are satisfied:

i) in the *capacitated* VRP (CVRP), there is a positive weight q_i at vertex v_i $(i = 2, \ldots, n)$ and all vehicles have the same capacity Q. Then the sum of weights of any route must not exceed Q;

ii) in the *distance constrained* VRP (DVRP), the total length of any route may not exceed a preset bound L;

iii) in the VRP *with time windows* (TWVRP), every vertex v_i must be visited within a specified time frame $[a_i, b_i]$ $(i = 2, \ldots, n)$. Note that waiting is allowed, i.e. a vehicle may arrive at v_i before a_i and wait until a_i to actually visit that vertex.

Comprehensive references on the TSP and on several types of VRPs can be found in Lawler et al. [9], Laporte and Nobert [8] and Golden and Assad [3]. For the TWVRP, see Desrochers et al. [2].

Several formulations have been suggested for the TSP. Let x_{ij} be a binary variable equal to 1 if and only if arc (v_i, v_j) is used in the solution. Then the Dantzig, Fulkerson and Johnson (DFJ) formulation [1] is:

(DFJ) $$\text{minimize} \sum_{i \neq j} c_{ij} x_{ij} \tag{1}$$

subject to

$$\sum_{\substack{j=1 \\ j \neq i}}^{n} x_{ij} = 1 \qquad (i = 1, \ldots, n) \tag{2}$$

$$\sum_{\substack{i=1 \\ i \neq j}}^{n} x_{ij} = 1 \qquad (j = 1, \ldots, n) \tag{3}$$

$$\sum_{i,j \in S} x_{ij} \leq |S| - 1 \qquad (S \subset V, |S| \geq 2) \qquad (4)$$

$$x_{ij} \in \{0, 1\} \qquad (i, j = 1, \ldots, n\,;\, i \neq j) \qquad (5)$$

Constraints (2) and (3) are degree constraints, and constraints (4) are subtour elimination constraints. Grötschel and Padberg [4] have proved that the latter constraints are *facets* of the polytope of feasible TSP solutions.

The Miller, Tucker and Zemlin (MTZ) formulation [10] reduces the number of subtour elimination constraints at the expense of extra free variables u_i $(i = 2, \ldots, n)$:

$$u_i - u_j - n x_{ij} \leq n - 1 \qquad (i, j = 2, \ldots, n\,;\, i \neq j) \qquad (6)$$

Note that it is valid to restrict the u_i's to take values in $[1, n-1]$. There is then only one solution corresponding to a given feasible tour. It is well known that this formulation produces a weak LP relaxation [6],[11]. This can be explained by the fact that the MTZ subtour elimination constraints are not facets of the polytope of feasible solutions; they are even faces of relatively low dimension.

In spite of their relative weakness, these constraints offer some advantages: they generalize quite easily to a number of VRP formulations and, in a subtour elimination constraint relaxation algorithm, it is straightforward to identify which of these (generalized) constraints are violated. Note that the DFJ subtour elimination constraints can also easily be extended to the CVRP [7]. However, their generalization to the DVRP is not so easy [7] and no such extension is known for the TWVRP. Another advantage of the MTZ subtour elimination constraints is that they can be incorporated in other type of formulations, in conjunction with stronger types of constraints.

The object of this paper is to strengthen the MTZ subtour elimination constraints for the TSP (section 2) and to generalize these stronger constraints to a variety of VRPs (section 3).

2. Lifting the MTZ subtour elimination constraints for the TSP.

The MTZ subtour elimination constraints for the TSP can be strengthened by using a lifting technique [12]. More specifically, the original MTZ inequality (6) for arc (v_i, v_j) $(i, j \geq 2)$, can be rewritten as

$$u_i - u_j + n x_{ij} + \alpha_{ji} x_{ji} \leq n - 1, \qquad (7)$$

where currently $\alpha_{ji} = 0$ and $u_i, u_j \in [1, n-1]$. The lifting process computes the largest possible value for α_{ji} so that (7) remains a valid inequality. There are two cases: $x_{ji} = 0$ and $x_{ji} = 1$.

Case 1: $x_{ji} = 0$. Then $u_i - u_j + nx_{ij} + \alpha_{ji}x_{ji} \leq n-1$ is verified for any α_{ji} ;

Case 2: $x_{ji} = 1$. This implies $x_{ij} = 0$ (for $n > 2$) and $u_j + 1 = u_i$ so that $\alpha_{ji} \leq n-2$.

It should be noted that (7) cannot be lifted further since for any other variable $x_{k\ell}$, there is a tight solution to the inequality if $x_{k\ell} = 0$ and if $x_{k\ell} = 1$.

The same lifting process can be applied to the bounds on the u_i variables :

$$1 \leq u_i \leq n-1 \qquad\qquad i = 2, \ldots, n \qquad (8)$$

In the case of the lower bound $u_i \geq 1$, we can apply the lifting process to the x_{ji} variables ($j = 2, \ldots, n$, $j \neq i$) to obtain the following augmented inequality :

$$u_i \geq 1 + \sum_{\substack{j=2 \\ j \neq i}}^{n} x_{ji}.$$

The same procedure is then applied to variables x_{i1} to yield :

$$u_i \geq 1 + (n-3)x_{i1} + \sum_{\substack{j=2 \\ j \neq i}}^{n} x_{ji} \qquad (9)$$

and finally, a similar valid inequality is obtained by lifting the upper bounds :

$$u_i \leq n - 1 - (n-3)x_{1i} - \sum_{\substack{j=2 \\ j \neq i}}^{n} x_{ij}. \qquad (10)$$

To gain some insight into whether or not the lifted MTZ formulation provides some improvement over the original MTZ formulation or the assignment bound, we generated TSPs from three classes: Asymmetric random cost, Symmetric random cost and Symmetric Euclidean cost. For each class, ten 50-city problems were solved. In each case, three values were computed :

1) ASS : the value of the assignment bound ((1)–(3));

2) MTZ : the solution value obtained by incorporating all violated MTZ constraints (6);

3) MTZ$^+$: the solution value obtained by incorporating all violated lifted MTZ constraints (7), (9) and (10).

The ratios MTZ/ASS and MTZ$^+$/ASS obtained for each of the thirty problems are given in Table 1. It can be noted that using the lifted MTZ constraints improves on the relaxation obtained by using the MTZ constraints by an amount varying between 11% and 36% depending on the type of problem considered. The relatively low improvement obtained in the case of Asymmetrical random cost problems is due to the fact that in these problems, the ASS value is often close to the optimal value.

Asymmetric Random cost*			Symmetric Random cost*			Symmetric Euclidean cost*		
Pr.#	MTZ/ASS	MTZ$^+$/ASS	Pr.#	MTZ/ASS	MTZ$^+$/ASS	Pr.#	MTZ/ASS	MTZ$^+$/ASS
1	1.02	1.15	1	1.01	1.21	1	1.01	1.20
2	1.01	1.07	2	1.02	1.52	2	1.00	1.10
3	1.01	1.03	3	1.01	1.43	3	1.01	1.32
4	1.01	1.13	4	1.01	1.34	4	1.01	1.27
5	1.02	1.02	5	1.00	1.28	5	1.00	1.13
6	1.01	1.14	6	1.01	1.46	6	1.01	1.25
7	1.01	1.10	7	1.01	1.26	7	1.01	1.16
8	1.01	1.33	8	1.01	1.40	8	1.01	1.30
9	1.02	1.07	9	1.01	1.41	9	1.01	1.20
10	1.02	1.28	10	1.01	1.48	10	1.01	1.23
Average % improvement: 11.7%			Average % improvement: 36.7%			Average % improvement: 20.7%		
*c_{ij} uniformly distributed in [0..100]			*c_{ij} uniformly distributed in [0..100]			*$c_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$ *x_i and y_i uniformly distributed in [0..100]		

TABLE 1

Comparison of ASS, MTZ and MTZ$^+$

3. Extending the lifted MTZ constraints to three types of VRPs.

The improvements presented in section 2 can be extended to several types of VRPs.

3.1 Capacitated vehicle routing problem.

The original MTZ subtour elimination constraints can easily be adapted to the CVRP (see, for example, Kulkarni and Bhave [5]):

$$u_i - u_j + Qx_{ij} + \alpha_{ji}x_{ji} \leq Q - q_i \quad (i, j = 2, \ldots, n\,;\, i \neq j\,;\, q_i + q_j \leq Q) \tag{11}$$

and

$$0 \leq u_i \leq Q - q_i \qquad (i = 2, \ldots, n), \qquad (12)$$

where α_{ji} is currently 0. Applying once more the lifting process yields $\alpha_{ji} = Q - q_i - q_j$. Further improvements are still possible. For example, for any vertex v_k such that $q_i + q_j + q_k > Q$, a stronger constraint of the form

$$u_i - u_j + Qx_{ij} + \alpha_{ji}x_{ji} + \alpha_{ik}x_{ik} + \alpha_{kj}x_{kj} + \alpha_{ki}x_{ki} + \alpha_{jk}x_{jk} \leq Q - q_i \qquad (13)$$

can be generated. Here, the α coefficients are set equal to $\alpha_{ji} = Q - q_i - q_j$, $\alpha_{ik} = \alpha_{kj} = q_k$ and $\alpha_{ki} = \alpha_{jk} = 0$. To illustrate, consider α_{ik}. Setting $x_{ik} = 1$ implies $x_{ki} = x_{ji} = x_{ij} = x_{jk} = x_{kj} = 0$, and (13) can be rewritten as

$$\alpha_{ik} \leq Q - q_i - u_i + u_j.$$

Since $u_j \geq 0$ and $u_i \leq Q - q_i - q_k$, it follows that $\alpha_{ik} \leq q_k$. A similar lifting process can be applied to constraints (12). The general bounding constraints are

$$u_i \geq \sum_{\substack{j \neq i \\ j \neq 1}} \alpha_{ji}x_{ji} \qquad (14)$$

and

$$u_i \leq Q - q_i - \alpha_{1i}x_{1i} - \sum_{\substack{j \neq i \\ j \neq 1}} \alpha_{ij}x_{ij}. \qquad (15)$$

The α coefficients are then $\alpha_{ji} = q_j$ in (14), $\alpha_{1i} = Q - \max_{j \neq i} \{q_j\} - q_i$ and $\alpha_{ij} = q_i$ in (15). Note that these values, with $Q = n - 1$ and $q_i = 1$, are consistent with (9) and (10).

3.2 Distance constrained vehicle routing problem.

Now consider the DVRP and define s_i, the length of a shortest path from v_1 to v_i and t_i, the length of a shortest path from v_i to v_1. Observe that $x_{ij} = 0$ whenever $s_i + c_{ij} + t_j > L$. We first prove that the following constraint system is valid :

$$u_i - u_j + Mx_{ij} \leq M - c_{ij} \qquad (i, j = 2, \ldots, n\,;\, i \neq j), \qquad (16)$$

$$u_i \leq L - t_i - (c_{i1} - t_i)x_{i1} \qquad (i = 2, \ldots, n) \qquad (17)$$

and

$$u_i \geq s_i + (c_{1i} - s_i)x_{1i} \qquad (i = 2, \ldots, n), \qquad (18)$$

where $M \geq \max_{i,j} \{L - s_j - t_i + c_{ij}\}$ in order not to eliminate feasible solutions when $x_{ij} = 0$.

To prove that constraints (16) are valid, we must show that they eliminate subtours disconnected from v_1, and subtours connected to v_1 but of total length exceeding L. In the first case, consider a subtour $(v_{i_1}, \ldots, v_{i_k})$ where $v_{i_1}, \ldots, v_{i_k} \neq v_1$. Summing up the corresponding constraints yields :

$$\sum_{\ell=1}^{k} \left(u_{i_\ell} - u_{i_{\ell+1}} + M x_{i_\ell i_{\ell+1}} - M + c_{i_\ell i_{\ell+1}} \right) \leq 0,$$

where $i_{k+1} \equiv i_1$. This reduces to

$$\sum_{\ell=1}^{k} \left[M \left(x_{i_\ell i_{\ell+1}} - 1 \right) + c_{i_\ell i_{\ell+1}} \right] \leq 0$$

and to

$$\sum_{\ell=1}^{k} c_{i_\ell i_{\ell+1}} \leq 0 \,,$$

a contradiction, unless G contains subtours of zero length. Now consider a subtour $(v_{i_1} \equiv v_1, v_{i_2}, \ldots, v_{i_k})$. Again, summing up constraints (16) for arcs $(v_{i_\ell}, v_{i_{\ell+1}})$, $\ell = 2, \ldots, k-1$ gives :

$$\sum_{\ell=2}^{k-1} \left(u_{i_\ell} - u_{i_{\ell+1}} + M x_{i_\ell i_{\ell+1}} - M + c_{i_\ell i_{\ell+1}} \right) \leq 0,$$

$$u_{i_2} - u_{i_k} + \sum_{\ell=2}^{k-1} c_{i_\ell i_{\ell+1}} \leq 0$$

and

$$\sum_{\ell=1}^{k-1} c_{i_\ell i_{\ell+1}} \leq u_{i_2} + \sum_{\ell=2}^{k-1} x_{i_\ell i_{\ell+1}} \leq u_{i_k}$$

since $u_{i_2} \geq c_{1 i_2}$. Finally, this reduces to

$$\sum_{\ell=1}^{k} c_{i_\ell i_{\ell+1}} \leq L$$

since $u_{i_k} \leq L - c_{i_k 1}$. □

Constraints (16) can be augmented by adding an extra term $\alpha_{ji} x_{ji}$:

$$u_i - u_j + M x_{ij} + \alpha_{ji} x_{ji} \leq M - c_{ij} \qquad (i, j = 2, \ldots, n \,;\, i \neq j) \tag{19}$$

where α_{ji} can be as large as $M - c_{ij} - c_{ji}$. Further improvements can still be obtained by using the fact that some arc combinations are infeasible and, as in Laporte *et al.* [7], several cases are

possible. For the sake of brevity, we provide only one example. Consider two arcs (v_k, v_i) and (v_i, v_j) such that $s_k + c_{ki} + c_{ij} + t_j > L$. Constraints (16) then become :

$$u_i - u_j + Mx_{ij} + (M - c_{ij} - c_{ji})x_{ji} + \alpha_{ki}x_{ki} \leq M - c_{ij} \qquad (i, j = 2, \ldots, n\,;\, i \neq j), \qquad (20)$$

where $\alpha_{ki} \leq M - L + s_j + t_i - c_{ij}$. As above, the lower and upper bound constraints on u_i can also be lifted to yield :

$$u_i \geq s_i + \sum_{\substack{j=1 \\ j \neq i}}^{n} (s_j + c_{ji} - s_i)x_{ji} \qquad (i = 2, \ldots, n) \qquad (21)$$

and

$$u_i \leq L - t_i - \sum_{\substack{j=1 \\ j \neq i}}^{n} (t_j + c_{ij} - t_i)x_{ij} - (L - c_{1i} - \max_{\substack{j \neq i \\ j \neq 1}} \{t_j + c_{ij}\})x_{i1} \; (i = 2, \ldots, n). \qquad (22)$$

3.3 Vehicle routing problem with time windows.

Subtour elimination constraints for the TWVRP can be written as :

$$u_i - u_j + Mx_{ij} \leq M - c_{ij} \qquad (i, j = 2, \ldots, n\,;\, i \neq j) \qquad (23)$$

and

$$a_i \leq u_i \leq b_i \qquad (i = 2, \ldots, n), \qquad (24)$$

where $[a_i, b_i]$ is the time window for vertex v_i and $M \geq \max_{i,j} \{b_i - a_j + c_{ij}\}$.

Because of time window constraints, some arcs (v_i, v_j) are infeasible. The following lifting process for constraints (23) only applies if (v_i, v_j) and (v_j, v_i) are both defined. The constraint is then

$$u_i - u_j + Mx_{ij} + \alpha_{ji}x_{ji} \leq M - c_{ij} \qquad (i, j = 2, \ldots, n\,;\, i \neq j). \qquad (25)$$

where several cases must be considered for the determination of α_{ji}.

Case 1 : $a_j + c_{ji} \geq a_i$. Then, $\alpha_{ji} \leq M - c_{ij} - c_{ji}$.

Case 2 : $a_j + c_{ji} < a_i$ and $b_j + c_{ji} \geq a_i$. Then, $\alpha_{ji} \leq M - c_{ij} + \min\{-c_{ji}\,,\, b_j - a_i\}$.

Case 3 : $b_j + c_{ji} < a_i$. Then, as in case 2, $\alpha_{ji} \leq M - c_{ij} + \min\{-c_{ji}\,,\, b_j - a_i\}$.

It should be noted that by definition of case 1, $-c_{ji} \leq a_j - a_i \leq b_j - a_i$ and thus, in general,

$$\alpha_{ji} \leq M - c_{ij} + \min\{-c_{ji}, b_j - a_i\}.$$

As in the two previous sub-sections, it is possible to further improve constraints (25) by taking into account infeasible arc combinations. Again, for the sake of brevity, we give only one example. Consider two arcs (v_k, v_i) and (v_i, v_j) such that $a_k + c_{ki} + c_{ij} > b_j$. Constraints (16) then become :

$$u_i - u_j + M x_{ij} + \Big(M - c_{ij} + \min\{-c_{ji},\, b_j - a_i\}\Big) x_{ji} + \alpha_{ki} x_{ki} \leq M - c_{ij}$$
$$(i, j = 2, \ldots, n\,;\, i \neq j), \qquad (26)$$

where $\alpha_{ki} \leq M - c_{ij} - \min\{b_k + c_{ki},\, b_i\} + a_j$.

Finally, the lower and upper bound constraints on u_i can be lifted to yield:

$$u_i \geq a_i + \sum_{\substack{j=1 \\ j \neq i}}^{n} \Big(\max\{0,\, a_j - a_i + c_{ji}\}\Big) x_{ji} \qquad (27)$$

and

$$u_i \leq b_i + \sum_{\substack{j=1 \\ j \neq i}}^{n} \Big(\max\{0,\, b_i - b_j + c_{ij}\}\Big) x_{ij}. \qquad (28)$$

4. Conclusion

We have shown that the Miller-Tucker-Zemlin subtour elimination constraints for the TSP can be improved and preliminary tests indicate that the gain is non-negligible. Furthermore this improvement can be used in the extensions to other routing problems of the MTZ formulation. Here lies most of the interest for these formulations.

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